Tropical analysis of plurisubharmonic singularities

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Abstract

Tropical structures appear naturally in investigation of singularities of plurisubharmonic functions. We show that standard characteristics of the singularities can be viewed as tropicalizations of certain notions from commutative algebra. In turn, such a consideration gives a tool for studying the singularities. In addition, we show how the notion of Newton polyhedron and its generalizations come into the picture as a result of the tropicalization.

1 Introduction

We recall that a semiring S with an addition $\oplus$ and multiplication $\otimes$ is called idempotent if $s \oplus s = s$ for any $s \in S$. When S is a subset of the extended real line, $a \oplus b = \max\{a, b\}$ (or $\min\{a, b\}$) and $a \otimes b = a + b$, such a semiring is usually called tropical. For basics on idempotent/tropical structures, see, e.g., [18] and the bibliography therein.

In this note, we consider certain tropical semirings arising naturally in multidimensional complex analysis. This starts with a simple observation that a basic object of pluripotential theory – plurisubharmonic functions – can be viewed as Maslov's dequantization of analytic functions (a basic object of the whole complex analysis). To detect a tropical structure, we need to pass from the world of complex values to the real one. This makes sense in consideration of asymptotic behavior of absolute values $|f(z)|$ of analytic functions f when z approaches either the zero set of f or infinity. Here we will be concerned with the former (local) situation, which invokes investigation of singularities of plurisubharmonic functions and corresponding tropical semirings.

Standard characteristics of singularities of plurisubharmonic functions are thus "tropicalizations" of notions from commutative algebra and can be viewed as functionals on the corresponding tropical semiring. Central role here is played by tropically linear functionals (i.e., additive and multiplicative with respect to the tropical operations and homogeneous with respect to the usual multiplication by positive constants). A problem of description for such functionals is posed. On the other hand, a larger class of the functionals, just tropically additive and positive homogeneous, is described, and a relation between these two classes is established. A

few other problems are formulated as well. In addition, we show that the linear functionals can be thought of as "tropicalizations" of valuations on the local ring of germs of analytic functions as well.

Another way of using Maslov's dequantization is to perform it on the arguments of the functions, which moves us from functions on complex manifolds to functions on $\mathbb{R}^n$. This results in a notion of local indicator, introduced from a different point of view in [17]. The semiring of plurisubharmonic singularities maps to a tropical semiring of the indicators, and the latter turns out to be isomorphic to an idempotent semiring of complete convex subsets of $\mathbb{R}^n_+ = \{t \in \mathbb{R}^n : t_i > 0, \ i = 1, \ldots, n\}$. Going this way, the notion of Newton polyhedron comes naturally into the picture, together with generalizations of famous Kushnirenko's and Bernstein's results on bounds for multiplicities of holomorphic mappings in terms of (mixed) covolumes of the polyhedra.

Most of the results, except for those in Section 5, are obtained in [21]–[25], so we do not present their proofs here and just put them into the context of tropical mathematics. The proofs of the statements from Section 5 are sketched.

2 Plurisubharmonic singularities

An upper semicontinuous, real-valued function u on a complex manifold M is called *plurisubharmonic* (*psh*) if for every holomorphic mapping λ from the unit disk $\mathbb{D}$ into M, the function $u \circ \lambda$ is subharmonic (which means that for every point $\zeta \in \mathbb{D}$,

$$(u \circ \lambda)(\zeta) \le \frac{1}{2\pi} \int_0^{2\pi} (u \circ \lambda)(\zeta + re^{i\theta})\, d\theta$$

for all $r < 1 - |\zeta|$). A basic example is $u = c \log |f|$ with $c > 0$ and a function f analytic on M. Moreover, as follows from Bremermann's theorem [3], every psh function on a pseudoconvex domain $\omega \subset M$ belongs to the closure (in $L^1_{loc}(\omega)$) of the set of functions $\{\sup_\alpha c_\alpha \log |f_\alpha|\}$. For standard facts on psh functions, see, e.g., [12], [16], [27].

Let $\mathcal{O}_M$ be the ring of analytic functions f on M. The transformation $f \mapsto \log |f|$ maps it to the cone $\mathrm{PSH}(M)$ of psh functions on M, and the ring operations on $\mathcal{O}_M$ induce a natural tropical structure on $\mathrm{PSH}(M)$ with the addition

$$u \check{\oplus} v := \max\{u, v\},$$

which is based on Maslov's dequantization

$$f + g \mapsto \frac{1}{N} \log |f^N + g^N| \longrightarrow \max\{\log |f|, \log |g|\} \quad \text{as } N \to \infty,$$

and multiplication

$$u \otimes v := u + v$$

(simply by $fg \mapsto \log|fg| = \log|f| \otimes \log|g|$). Thus $\mathrm{PSH}(M)$ becomes a tropical semiring, closed under (usual) multiplication by positive constants. (We use the symbol $\check{\oplus}$ instead of $\oplus$ in order to emphasize that it is the max-addition; later on we will need another idempotent addition, $a\hat{\oplus}b = \min\{a,b\}$.) The neutral element (tropical 0) is $u \equiv -\infty$, and the unit (tropical 1) is $u \equiv 0$.

From now on, we restrict ourselves to local considerations, so in the sequel we deal with functions defined near $0 \in \mathbb{C}^n$. Let $\mathcal{O}_0$ denote the ring of germs of analytic functions at 0, and let $\mathfrak{m}_0 = \{f \in \mathcal{O}_0 : f(0) = 0\}$ be its maximal ideal. The above log-transformation sends $\mathcal{O}_0$ to the corresponding tropical semiring PSHG_0 of germs of psh functions. We will say that a psh germ u is *singular* at 0 if u is not bounded (below) in any neighbourhood of 0. For functions $u = \log|f|$ this means $f \in \mathfrak{m}_0$; asymptotic behaviour of arbitrary psh functions can be much more complicated (it may even happen that $u(0) > -\infty$).

A partial order on PSHG_0 is given as follows:

$$u \preceq v \Leftrightarrow u(z) \le v(z) + O(1), \quad z \to 0,$$

which leads to the equivalence relation $u \sim v$ if $u(z) = v(z) + O(1)$. The equivalence class $\mathrm{cl}(u)$ of u is called the *plurisubharmonic singularity* of u (in [29], a closely related object was introduced under the name "standard singularity"). The collection of psh singularities $\mathrm{PSHS}_0 = \mathrm{PSHG}_0/\sim$ has the same tropical structure $\{\check{\oplus}, \otimes\}$ and the partial order $\mathrm{cl}(u) \le \mathrm{cl}(v)$ if $u \preceq v$. The neutral element here is still $u \equiv -\infty$, while the unit is represented by any nonsingular germ.

Psh singularities form a cone with respect to the usual multiplication by positive numbers. Finally, they are endowed with the topology where $\mathrm{cl}(u_j) \to \mathrm{cl}(u)$ if there exists a neighbourhood ω of 0 and psh functions $v_j \in \mathrm{cl}(u_j)$, $v \in \mathrm{cl}(u)$ such that $v, v_j \in \mathrm{PSH}(\omega)$ and $v_j \to v$ in $L^1(\omega)$. By abusing the notation, we will write occasionally u for $\mathrm{cl}(u)$.

3 Characteristics of singularities

1. A fundamental characteristic of an analytic germ $f \in \mathfrak{m}_0$ is its multiplicity (vanishing order) m_f: if $f = \sum P_j$ is the expansion of f in homogeneous polynomials, $P_j(tz) = t^j P(z)$, then $m_f = \min\{j : P_j \not\equiv 0\}$.

The corresponding basic characteristic of singularity of $u \in \mathrm{PSHG}_0$ is its *Lelong number*

$$\nu(u) = \lim_{t\to-\infty} t^{-1} M(u,t) = \liminf_{z\to 0} \frac{u(z)}{\log|z|} = dd^c u \wedge (dd^c \log|z|)^{n-1}(0);$$

here $M(u,t)$ is the mean value of u over the sphere $\{|z| = e^t\}$; $d = \partial + \bar{\partial}$, $d^c = (\partial - \bar{\partial})/2\pi i$. If $f \in \mathfrak{m}_0$, then $\nu(\log|f|) = m_f$. This characteristic of singularity gives important information on the asymptotics: $u(z) \le \nu(u)\log|z| + O(1)$.

Since $\nu(v) = \nu(u)$ for all $v \in \mathrm{cl}(u)$, Lelong number can be considered as a functional on PSHS_0 with values in the tropical semiring $\mathbb{R}_+(\min, +)$ of non-negative real numbers with the operations

$$x \hat{\oplus} y = \min\{x, y\} \quad \text{and} \quad x \otimes y = x + y.$$

As such, it is
(i) positive homogeneous: $\nu(cu) = c\,\nu(u)$ for all $c > 0$,
(ii) additive: $\nu(u \check{\oplus} v) = \nu(u) \hat{\oplus} \nu(v)$,
(iii) multiplicative: $\nu(u \otimes v) = \nu(u) \otimes \nu(v)$, and
(iv) upper semicontinuous: $\nu(u) \geq \limsup \nu(u_j)$ if $u_j \to u$.

2. Lelong numbers are independent of the choice of coordinates (Siu's theorem). Let us now fix a coordinate system centered at 0. The *directional Lelong number* of $u \in \mathrm{PSHG}_0$ in a direction $a \in \mathbb{R}^n_+$ (introduced by C. Kiselman [10]) is

$$\nu(u, a) = \lim_{t \to -\infty} t^{-1} M(u, ta) = \liminf_{z \to 0} \frac{u(z)}{\phi_a(z)}, \tag{1}$$

where $M(u, ta)$ is the mean value of u over the distinguished boundary of the polydisk $\{|z_k| < \exp(ta_k)\}$ and

$$\phi_a(z) = \check{\oplus}_k \, a_k^{-1} \log |z_k|. \tag{2}$$

Since its value is constant on $cl(u)$, it is well defined on PSHS_0. The functional has the same properties (i)–(iv), and the collection $\{\nu(u, a)\}_a$ gives refined information on the singularity u; in particular, $\nu(u) = \nu(u, (1, \ldots, 1))$.

For polynomilas or, more generally, analytic functions $f = \sum c_J z^J \in \mathfrak{m}_0$, it can be computed as

$$\nu(\log |f|, a) = \hat{\oplus}\{\langle a, J\rangle : c_J \neq 0\},$$

the expression in the right-hand side being known in number theory as the *index* of f with respect to the weight a [14].

3. A general notion of Lelong number with respect to a plurisubharmonic weight was introduced by J.-P. Demailly [4] (concerning the complex Monge-Ampère operator $(dd^c)^n$, the reader can consult [12] and [6]). Let $\varphi \in \mathrm{PSHG}_0$ be continuous and locally bounded outside 0. Then the mixed Monge–Ampère current $dd^c u \wedge (dd^c \varphi)^{n-1}$ is well defined for any psh function u and is equivalent to a positive Borel measure. Its mass at 0,

$$\nu(u, \varphi) = dd^c u \wedge (dd^c \varphi)^{n-1}(\{0\}), \tag{3}$$

is the *generalized*, or *weighted, Lelong number* of u with respect to the weight φ. By Demailly's comparison theorem, it is constant on $\mathrm{cl}(u)$ and thus defines a functional on PSHS_0. It still has the above properties (i), (iii), and (iv), however in general is only subadditive: $\nu(u \check{\oplus} v, \varphi) \leq \nu(u, \varphi) \hat{\oplus} \nu(v, \varphi)$.

4. One more characteristic, the *integrability index*

$$\lambda_u = \inf\{\lambda > 0 : e^{-u/\lambda} \in L^2_{loc}\}, \tag{4}$$

is both subadditive and submultiplicative, and it is also upper semicontinuous [7]. If $f = (f_1, \dots, f_m) \in \mathfrak{m}_0^m$, the value $\lambda_{\log|f|}$ is known as the *Arnold multiplicity* of the ideal $\mathcal{I}$ generated by f_j, and

$$lc(\mathcal{I}) = \lambda^{-1}_{\log|f|} \tag{5}$$

is the *log canonical threshold* of $\mathcal{I}$.

4 Additive functionals

Another generalization of the notion of Lelong number was introduced in [25]. Let $\varphi \in \mathrm{PSHG}_0$, singular at 0, be locally bounded and *maximal* outside 0 (that is, satisfies $(dd^c\varphi)^n = 0$ on a punctured neighbourhood of 0); the collection of all such germs (*maximal weights*) will be denoted by MW_0. An important example of such a weight is $\varphi = \log|F|$ for an equidimensional holomorphic mapping F with isolated zero at the origin.

For $u \in \mathrm{PSHG}_0$ (or $u \in \mathrm{PSHS}_0$), its *type relative to* $\varphi \in \mathrm{MW}_0$ is defined as

$$\sigma(u, \varphi) = \liminf_{z \to 0} \frac{u(z)}{\varphi(z)}.$$

When $u = \log|f|$ and $\varphi = \log|F|$, the relative type $\sigma(u, \varphi)$ equals the value $\bar{\nu}_{\mathcal{I}}(f)$ considered in [15], $\mathcal{I}$ being the ideal in $\mathcal{O}_0$ generated by the components of the mapping F. For the directional weights ϕ_a (2),

$$\sigma(u, \phi_a) = \nu(u, a) = a_1 \dots a_n \, \nu(u, \phi_a).$$

Since the function $t \mapsto \sup\{u(x) : \ \varphi(x) < t\}$ is convex, one has the relation

$$u \preceq \sigma(u, \varphi)\varphi.$$

Given $\varphi \in \mathrm{MW}_0$, the functional $\sigma(\cdot, \varphi) : \mathrm{PSHS}_0 \to [0, +\infty]$ is positive homogeneous, additive, supermultiplicative, and upper semicontinuous. Actually, relative types give a general form for all "reasonable" additive functionals on PSHS_0:

Theorem 4.1 [25] *Let a functional* $\sigma : \mathrm{PSHS}_0 \to [0, \infty]$ *be such that*

1) $\sigma(cu) = c\,\sigma(u)$ *for all* $c > 0$*;*

2) $\sigma(\check{\bigoplus} u_k) = \hat{\bigoplus} \sigma(u_k)$, $k = 1, 2$*;*

3) if $u_j \to u$*, then* $\limsup \sigma(u_j) \le \sigma(u)$*;*

4) $\sigma(\log|z|) = 1$*;*

5) $\sigma(u) < \infty$ *if* $u \not\equiv -\infty$*.*

Then there exists a weight $\varphi \in \mathrm{MW}_0$ *such that* $\sigma(u) = \sigma(u, \varphi)$ *for every singularity* $u \in \mathrm{PSHS}_0$*. The representation is essentially unique: if two maximal weights* φ *and* ψ *represent* σ*, then* $\mathrm{cl}(\varphi) = \mathrm{cl}(\psi)$*.*

In particular, such a functional σ is always supermultiplicative; if $n = 1$, it is multiplicative (equal to the mass of the Riesz measure $\frac{1}{2\pi}\Delta u$ at 0).

The function $\varphi \in \mathrm{MW}_0$ from the theorem can be constructed by the Perron method (as a corresponding Green function): given a bounded hyperconvex neighbourhood Ω of 0, φ is the upper envelope of all negative psh functions v in Ω such that $\sigma(v) \geq 1$.

5 Additivity vs linearity

A functional on PSHS_0 is *(tropically) linear* if it is both additive and multiplicative; we will also assume it to be positive homogeneous and upper semicontinuous. The collection of all such functionals will be denoted by L_0.

An example of linear functional is $u \mapsto \nu(u \circ \mu, p)$, the Lelong number of the pullback of u by a holomorphic mapping μ at $p \in \mu^{-1}(0)$. Another example are the directional Lelong numbers $\nu(u, a)$ defined by (1).

In a usual vector space, every convex function is an upper envelope of affine ones. In our situation, any tropically additive functional is superadditive in the usual sense, while tropically linear ones are additive. This raises the following question.

Problem 1. Is it true that all tropically additive functionals on PSHS_0 can be represented as lower envelopes of tropically linear ones?

We can answer the question for the functionals generated by weights that can be approximated by multiples of logarithms of moduli of holomorphic mappings. First, let us take $\varphi = \log|F| \in \mathrm{MW}_0$, where F is an equidimensional holomorphic mapping with isolated zero at the origin. By the Hironaka desingularization theorem, there exists a "log resolution" for the mapping F, i.e., a proper holomorphic mapping μ of a manifold X to a neighborhood U of 0, that is an isomorphism between $X \setminus \mu^{-1}(0)$ and $U \setminus \{0\}$, such that $\mu^{-1}(0)$ is a normal crossing divisor with components $E_1, \ldots, E_N$, and in local coordinates centered at a generic point p of a nonempty intersection $E_I = \cap_{i \in I} E_i$, $I \subset \{1, \ldots, N\}$,

$$(F \circ \mu)(x) = h(x) \prod_{i \in I} x_i^{m_i}$$

with $h(0) \neq 0$. Then for any $u \in \mathrm{PSHS}_0$, one has

$$\sigma(u, \varphi) = \min\{\nu_{I, m_I}(u \circ \mu) : E_I \neq \emptyset\},$$

where

$$\nu_{I,m_I}(u\circ\mu) = \liminf_{x\to 0} \frac{(u\circ\mu)(x)}{\sum_{i\in I} m_i \log|x_i|}$$

at a generic point of $p \in E_I$. It is then easy to see that

$$\nu_{I,m_I}(u\circ\mu) \geq \min_{i\in I} \nu_{i,m_i}(u\circ\mu) = \min_{i\in I} m_i^{-1}\nu_i(u\circ\mu),$$

where $\nu_i(u\circ\mu)$ is the Lelong number of $u\circ\mu$ at a generic point of E_i, which is a linear functional. This gives us the following result (proved for ideals $\mathcal{I}\subset\mathcal{O}_0$ in [15]).

Theorem 5.1 *For any weight* $\log|F| \in \mathrm{MW}_0$ *there exist finitely many functionals* $l_j \in \mathrm{L}_0$ *such that* $\sigma(u, \log|F|) = \min_j l_j(u)$ *for every* $u \in \mathrm{PSHS}_0$*; in other words,*

$$\sigma(u, \log|F|) = \hat{\oplus}_j\, l_j(u), \quad u \in \mathrm{PSHS}_0.$$

Furthermore, let us say that a function $\varphi \in \mathrm{PSHG}_0$ has *asymptotically analytic singularity* if for any $\epsilon > 0$ there exist positive integers p and q, a constant $C > 0$, a neighbourhood U of 0, and a holomorphic mapping $f : U \to \mathbb{C}^p$ such that

$$(1+\epsilon)\varphi(z) - C \leq q^{-1}\log|f(z)| \leq (1-\epsilon)\varphi(z) + C, \quad z \in U. \tag{6}$$

It can be easily shown that any weight $\varphi \in \mathrm{MW}_0$ with asymptotically analytic singularity has a continuous representative $\psi \in cl(\varphi)\cap\mathrm{MW}_0$ which can be approximated as in (6) with $p = n$ for all $\epsilon > 0$. By using Demailly's approximation theorem [5], it was shown in [2] that (6) holds if e^φ is Hölder continuous or, more generally, if φ is a *tame* weight, which means that there exists a constant $C_\varphi > 0$ such that for any $t > C_\varphi$ the condition $|f|\exp\{-t\varphi\} \in L^2_{loc}$ for a function $f \in \mathcal{O}_0$ implies $\sigma(\log|f|, \varphi) \geq t - C_\varphi$. (Actually, we are unaware of any example of maximal weight whose singularity is not asymptotically analytic.)

The following result is a direct consequence of Theorem 5.1; for tame weights it is essentially proved in [2].

Theorem 5.2 *If* $\varphi \in \mathrm{MW}_0$ *has asymptotically analytic singularity, then*

$$\sigma(u,\varphi) = \hat{\oplus}\,\{l(u) : \ l \in \mathrm{L}_0,\ l \geq \sigma(\cdot,\varphi)\}, \quad u \in \mathrm{PSHS}_0.$$

In view of Theorem 4.1, the following problems are natural.

Problem 2. Describe all $\varphi \in \mathrm{MW}_0$ such that the functional $\sigma(\cdot,\varphi) \in \mathrm{L}_0$.

Problem 3. What are functionals satisfying all the conditions of Theorem 4.1 except for the last one?

Problem 4. Does there exist a functional $\sigma \neq 0$ satisfying conditions 1)–3) and 5) of Theorem 4.1, such that $\sigma(\log|z|) = 0$?

Problem 5. What are multiplicative functionals on PSHS_0?

6 Relative types and valuations

For basics on valuation theory, we refer to [30]. Recall that a *valuation* on the analytic ring $\mathcal{O}_0$ is a nonconstant function $\mu:\ \mathcal{O}_0 \to [0,+\infty]$ such that

$$\mu(f_1f_2)=\mu(f_1)+\mu(f_2),\quad \mu(f_1+f_2)\ge \min\{\mu(f_1),\mu(f_2)\},\quad \mu(1)=0;$$

a valuation μ is *centered* if $\mu(f)>0$ for all $f\in\mathfrak{m}_0$.

Every $\varphi\in \mathrm{MW}_0$ generates a functional $\sigma_\varphi(f)=\sigma(\log|f|,\varphi)$ on $\mathcal{O}_0$ satisfying

$$\sigma_\varphi(f_1f_2)\ge\sigma_\varphi(f_1)+\sigma_\varphi(f_2),\quad \sigma_\varphi(f_1+f_2)\ge\min\{\sigma_\varphi(f_1),\sigma_\varphi(f_2)\},\quad \sigma_\varphi(1)=0.$$

(In [15] such functions are called *order functions*, and in [21] – *filtrations*.) If the relative type functional $\sigma(\cdot,\varphi)$ is multiplicative, then σ_φ is a valuation, centered if $\sigma(\log|z|,\varphi)>0$.

One can thus consider tropically linear functionals on PSHS_0 as tropicalizations of certain valuations on $\mathcal{O}_0$. For example, the (usual) Lelong number is the tropicalization of the multiplicity valuation m_f. The types relative to the directional weights ϕ_a (2) are multiplicative functionals on PSHS_0, and σ_{ϕ_a} (Kiselman's directional Lelong numbers) are monomial valuations on $\mathcal{O}_0$.

It was shown in [8] for $n=2$ and in [2] in the general case that an important class of valuations (*quasi-monomial* valuations) can be realized as σ_ϕ; all other centered valuations are limits of increasing sequences of the quasi-monomial ones. In addition, the Demailly's weighted Lelong number $\nu(\cdot,\varphi)$ (3) with a tame weight φ is an average of valuations [2].

7 Local indicators as Maslov's dequantizations

Singular psh germs appear as Maslov's dequantization of analytic functions. As indicated by constructions in tropical algebraic geometry [28], it is reasonable to perform a dequantization in the argument as well. This turns out to be equivalent to consideration of *local indicators*, a notion introduced in [17] by a completely different argument.

For a fixed coordinate system at 0, let $\nu(u,a)$ be the directional Lelong number of $u\in\mathrm{PSHS}_0$ in the direction $a\in\mathbb{R}^n_+$, see (1). Then the function

$$\psi_u(t)=-\nu(u,-t),\quad t\in\mathbb{R}^n_-=-\mathbb{R}^n_+,$$

is convex and increasing in each t_k, so $\psi_u(\log|z_1|,\dots,|z_n|)$ can be extended (in a unique way) to a function $\Psi_u(z)$ plurisubharmonic in the unit polydisk $\mathbb{D}^n$, the *local indicator* of u at 0 [17]. Note that the map $u\mapsto\Psi_u$ keeps the tropical structure:

$$\Psi_{u\check{\oplus}v}=\Psi_u\check{\oplus}\Psi_v,\quad \Psi_{u\otimes v}=\Psi_u\otimes\Psi_v,\quad \Psi_{cu}=c\,\Psi_u.$$

It is easy to see that the indicators have the log-homogeneity property

$$\Psi_u(z_1,\ldots,z_n)=\Psi_u(|z_1|,\ldots,|z_n|)=c^{-1}\Psi_u(|z_1|^c,\ldots,|z_n|^c)\quad \forall c>0. \tag{7}$$

In particular, this implies $(dd^c\,\Psi_u)^n=0$ on $\{\Psi_u>-\infty\}$, so if Ψ_u is locally bounded outside 0, then $\Psi_u\in \mathrm{MW}_0$,

$$(dd^c\,\Psi_u)^n=N_u\delta_0$$

for some $N_u\ge 0$, and $N_u=0$ if and only if $\Psi_u\equiv 0$ (δ_0 is Dirac's δ-function at 0).

The indicator can be viewed as the tangent (in the logarithmic coordinates) for the function u at 0 in the following sense.

Theorem 7.1 [21] *The indicator $\Psi_u(z)$ is a unique L^1_{loc}-limit of the functions*

$$T_m u(z)=m^{-1}u(z_1^m,\ldots,z_n^m),\quad m\to\infty. \tag{8}$$

In the tropical language, this means that for $f\in\mathfrak{m}_0$, the sublinear function $\psi_{\log|f|}(t)$ on $\mathbb{R}^n_-$ is just a Maslov's dequantization of f:

$$\psi_{\log|f|}(t)=\lim_{m\to\infty} m^{-1}\log|f(e^{m(t_1+i\theta_1)},\ldots,e^{m(t_n+i\theta_n)})|,$$

an interesting moment here being that the arguments become real by themselves.

The indicators are psh characteristics of psh singularities:

$$u(z)\le \Psi_u(z)+O(1); \tag{9}$$

if $\Psi\in L^\infty_{loc}(\mathbb{D}^n\setminus\{0\})$, then $\Psi_u\in\mathrm{MW}_0$ and the relative type $\sigma(u,\Psi_u)=1$. When u has isolated singularity at 0, this implies (by Demailly's comparison theorem [4]) the following relation between the Monge-Ampère measures:

$$(dd^c u)^n\ge (dd^c\,\Psi_u)^n=N_u\delta_0;$$

note that the measures $(dd^c\,T_m u)^n$ of $T_m u$ (8) need not converge to $(dd^c\,\Psi_u)^n$.

More generally, if for an n-tuple of psh functions u_k the current $dd^cu_1\wedge\ldots\wedge dd^cu_n$ is well defined near 0, then

$$dd^cu_1\wedge\ldots\wedge dd^cu_n\ge dd^c\Psi_{u_1}\wedge\ldots\wedge dd^c\Psi_{u_n}=N_{\{u_k\}}\delta_0.$$

In addition, relation (9) gives an upper bound for the integrability index λ_u (4),

$$\lambda_u\ge\lambda_{\Psi_u}; \tag{10}$$

unlike the situation with the Monge-Ampère measures, one in fact has $\lambda_{T_m u}\to\lambda_{\Psi_u}$ for the functions $T_m u$ defined by (8), which follows from a semicontinuity property for the integrability indices proved in [7].

In the case of a multicircled singularity $u(z)=u(|z_1|,\ldots,|z_n|)$, one has actually the equalities $(dd^cu)^n(0)=(dd^c\Psi_u)^n(0)$ (proved in [22]) and $\lambda_u=\lambda_{\Psi_u}$, which follows, by the same semicontinuity property, from the observation that in this case, Ψ_u is the upper envelope of negative psh functions v in $\mathbb{D}^n$ such that $v\le u+O(1)$ near 0.

8 Indicators and Newton polyhedra

Since Ψ_u is log-homogeneous, one can compute explicitly its Monge-Ampère mass and integrability index.

By transition from the psh function Ψ_u to the convex function

$$\psi_u(t) = \Psi_u(e^{t_1}, \ldots, e^{t_n}), \quad t \in \mathbb{R}^n_-, \tag{11}$$

and from the complex Monge-Ampère operator to the real one, we get a representation of the Monge-Ampère measures in terms of euclidian volumes. Let $\langle a, b\rangle$ stand for the scalar product in $\mathbb{R}^n$. The convex image ψ_u of the indicator Ψ_u coincides with the support function to the convex set

$$\Gamma_u = \{b \in \mathbb{R}^n_+ :\ \psi_u(t) \ge \langle b, t\rangle\ \forall t \in \mathbb{R}^n_-\} = \{b \in \mathbb{R}^n_+ :\ \nu(u, a) \le \langle a, b\rangle\ \forall a \in \mathbb{R}^n_+\},$$

that is,

$$\psi_u(t) = \sup\{\langle t, a\rangle :\ a \in \Gamma_u\}.$$

These transformations define an isomorphism between the semiring of the indicators Ψ and the semiring of complete convex subsets Γ of $\mathbb{R}^n_+$ (the completeness being in the sense $a \in \Gamma \Rightarrow a + \mathbb{R}^n_+ \in \Gamma$), endowed with the operations

$$\Gamma_1 \dot{\oplus} \Gamma_2 = \mathrm{conv}\,(\Gamma_1 \cup \Gamma_2)$$

(the convex hull of the union) and

$$\Gamma_1 \dot{\otimes} \Gamma_2 = \Gamma_1 + \Gamma_2 = \{a + b :\ a \in \Gamma_1,\ b \in \Gamma_2\}$$

(Minkowski's addition), and multiplication by positive scalars c. We get then

$$\Gamma_{u \check{\oplus} v} = \Gamma_u \dot{\oplus} \Gamma_v, \quad \Gamma_{u \otimes v} = \Gamma_u \dot{\otimes} \Gamma_v, \quad \Gamma_{c\,u} = c\,\Gamma_u.$$

Let $\mathrm{Covol}\,(\Gamma)$ denote the euclidian volume of $\mathbb{R}^n_+ \setminus \Gamma$.

Theorem 8.1 [21] *The residual Monge-Ampère mass of $u \in \mathrm{PSHG}_0$ with isolated singularity at 0 has the lower bound*

$$(dd^c u)^n(0) \ge (dd^c\,\Psi_u)^n(0) = n!\,\mathrm{Covol}\,(\Gamma_u). \tag{12}$$

Similarly, the mass of the mixed Monge-Ampère current $dd^c\Psi_{u_1} \wedge \ldots \wedge dd^c\Psi_{u_n}$ (when well defined) equals $n!\,\mathrm{Covol}\,(\Gamma_{u_1}, \ldots, \Gamma_{u_n})$, where $\mathrm{Covol}\,(A_1, \ldots, A_n)$ is an n-linear form on convex subsets of $\mathbb{R}^n_+$ such that $\mathrm{Covol}\,(A, \ldots, A) = \mathrm{Covol}\,(A)$. This gives the relation

$$dd^c u_1 \wedge \ldots \wedge dd^c u_n(0) \ge n!\,\mathrm{Covol}\,(\Gamma_{u_1}, \ldots, \Gamma_{u_n}),$$

provided the left-hand side is well defined.

If 0 is an isolated zero of a holomorphic mapping $F = (f_1, \dots, f_n)$, then its multiplicity equals $m_F = (dd^c \log|F|)^n(0) = dd^c \log|f_1| \wedge \ldots \wedge dd^c \log|f_n|(0)$ and the set $\Gamma_{\log|F|}$ is the convex hull of the union of the *Newton polyhedra*

$$\Gamma_{\log|f_j|} = \mathrm{conv}\{J + \mathbb{R}^n_+ : \; D^{(J)} f_j(0) \neq 0\}$$

of f_j at 0, $1 \le j \le n$. Therefore, Theorem 8.1 implies Kushnirenko's bound [13]

$$m_F \ge n! \,\mathrm{Covol}\,(\Gamma_{\log|F|}), \tag{13}$$

while the relation $m_F \ge n!\,\mathrm{Covol}\,(\Gamma_{\log|f_1|}, \dots, \Gamma_{\log|f_n|})$ is a modification of the local variant of D. Bernstein's theorem [1, Theorem 22.10]. As shown in [25], an equality in (13) is true if and only if $\log|F| = \Psi_{\log|F|} + O(1)$.

The class of all log-homogeneous psh weights Ψ is generated by the directional weights ϕ_a (2), in the sense that $\Psi(z) = \check{\bigoplus}\{\phi_a(z) : \phi_a \le \Psi\}$ and the relative type

$$\sigma(u, \Psi) = \hat{\bigoplus}\{\nu(u, a) : a \in A_\Psi\}, \tag{14}$$

where $A_\Psi = \{a \in \mathbb{R}^n_+ : \nu(\Psi, a) \ge 1\}$. Moreover, the generalized Lelong number with respect to any log-homogeneous weight can be represented in terms of the directional numbers, too:

Theorem 8.2 [22] *For each $\varphi \in \mathrm{MW}_0$ there exists a positive Borel measure γ_φ on the set A_φ such that*

$$\nu(u, \varphi) \ge \nu(u, \Psi_\varphi) = \int_{A_\varphi} \nu(u, a)\, d\gamma_\varphi(a), \quad u \in \mathrm{PSHS}_0. \tag{15}$$

Note that representation (15) for $\nu(u, \Psi)$ is a (tropically) multiplicative analogue of the additive representation (14) for $\sigma(u, \Psi)$.

The function ψ_u (11) can be considered as the restriction of the valuative transform of u (action of u on valuations [8], [2]) to the set of all monomial valuations. Although relations (12) and (15) are coarser than the corresponding estimates in terms of the valuative transforms from [8] and [2], they give bounds that can be explicitly computed (the measure γ_φ is defined constructively [22]). This reflects one of the benefits of using tropical mathematics.

Finally, a direct computation involving the function ψ_u (11) shows that the integrability index (4) for the indicator Ψ_u can be computed as $\lambda_{\Psi_u} = \sup\{\psi_u(t)/\sum t_j\}$. By (10), it gives the bound in terms of the directional numbers $\nu(u, a)$:

$$\lambda_u \ge \lambda_{\Psi_u} = \sup\{\nu(u, a) : \; \sum_k a_k = 1\},$$

with an equality in the case of multicircled singularity u (see the remark in the end of Section 7). This recovers [11, Thm. 5.8], which in turn implies a formula for the log canonical threshold (5) for monomial ideals proved independently in [9].

9 Related topics

The results on local indicators have global counterparts for psh functions of logarithmic growth in $\mathbb{C}^n$ (i.e., $\limsup_{|z|\to\infty} u(z)/\log|z| < \infty$, a basic example being $u = \log|P|$ for a polynomial mapping P), see [23] and [24]; they are also connected with the notion of *amoebas* of holomorphic functions [20]. Similar notions concerning Maslov's dequantization in $\mathbb{C}^n$ and generalized Newton polytops were also considered in [19].

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